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AN IMPROVED THEORY OF LONG WAVES ON THE WATER SURFACE†

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The Hamiltonian formalism is used to derive the equations of long waves on the surface of an infinite layer of water over a horizontal, smooth bottom, taking into account second-order terms with respect to small parameters of non-linearity and dispersion; in other words, the Boussinesq equations of shallow-water theory are improved. A non-linear evolution equation is derived for the elevation of the free surface and a transformation is obtained to convert it into one of the higher-order Korteweg–de Vries (KdV) equations. Single- and double-soliton solutions are used to demonstrate the special features of the behaviour of the waves described by the equation, which are a more accurate version of KdV solitons. © 1997 Elsevier Science Ltd. All rights reserved.

There have been several publications [1–5] on the problem of incorporating higher-order terms into the theory of long waves on a water surface. In the present paper we derive an evolution equation describing arbitrary long waves travelling in one direction and we reduce that equation to canonical form. This equation has a large class of solutions, for example, of the type of solitons or finite-band solutions that do not reduce to the description of the interaction of rapidly oscillating waves.

1. FORMULATION OF THE PROBLEM

We will consider the potential motion of a wave over an infinite horizontal bottom. The water is assumed to be an ideal incompressible homogeneous liquid and has a depth h in the undisturbed state. We introduce a Cartesian system of coordinates (x, y, z) with unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, with the z axis directed vertically upward and $z = 0$ the undisturbed surface of the water. Let $z = \eta(x, y, t)$ be the equation of the free surface. We will study long waves on the water surface, neglecting capillary and friction forces at the bottom.

The velocity of long waves on the water surface in the linear approximation is $\sqrt{(gh)}$, while for waves propagating in one direction the horizontal velocity is of the order of $\eta\sqrt{(gh)}/h$ [6]. These statements yield suitable scales of the required functions on which all further discussion will be based.

To analyse the dominant terms in the long-wave approximation, we define two small dimensionless parameters ε and μ as $\varepsilon = h^2/\lambda^2$, $\mu = a/h$, where λ is the characteristic wavelength and a the characteristic wave amplitude. We will assume that $\varepsilon \sim \mu$. We introduce dimensionless variables

$$x \to \lambda x,\ \ y \to \lambda y,\ \ z \to hz,\ \ \eta \to a\eta,\ \ t \to \frac{\lambda t}{\sqrt{gh}},\ \ \varphi \to \frac{ga\lambda\varphi}{\sqrt{gh}} \tag{1.1}$$

With these assumptions, the water motion is described by the following equations in dimensionless form

$$\varphi_{zz} + \varepsilon\Delta\varphi = 0,\ \ -1 < z < \mu\eta \tag{1.2}$$

$$\varphi_z = 0,\ \ z = -1 \tag{1.3}$$

$$\varphi_t + \frac{\mu}{2}(\nabla\varphi)^2 + \frac{\mu}{2\varepsilon}\varphi_z^2 + \eta = 0,\ \ \ z = \mu\eta \tag{1.4}$$

$$\eta_t + \mu\nabla\eta\cdot\nabla\varphi = \frac{\varphi_z}{\varepsilon},\ \ \ z = \mu\eta \tag{1.5}$$

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$$\left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}\right)$$

where φ is the velocity potential. Equation (1.2) follows from the Laplace equation for the velocity potential, Eq. (1.3) represents the impermeability of the bottom, and Eqs (1.4) and (1.5) are the Cauchy–Lagrange integral and kinematic condition on the free surface of the water, respectively. It can be verified that the order of φ_z in (1.4) and (1.5) is ε.

2. DERIVATION OF THE FUNDAMENTAL EQUATIONS

A solution of Eq. (1.2) satisfying boundary condition (1.3) will have the form of a power series [7, 8]

$$\varphi(x, y, z, t) = \sum_{n=0}^{\infty} \frac{(-1)^n \varepsilon^n (z+1)^{2n} \Delta^n f}{(2n)!} \tag{2.1}$$

where $f = \varphi(x, y, z = -1, t)$ is the value of the velocity potential of the water at the bottom. Henceforth we include in the equations only terms of orders 1, ε, μ, ε^2, μ^2 and $\varepsilon\mu$, that is, we limit ourselves to accuracy $O(\varepsilon^2, \mu^2)$. Denote the velocity potential at the free surface by Φ, that is, $\Phi = \varphi(x, y, z = \mu\eta, t)$. Then, by (2.1) we see that to within $O(\varepsilon^2, \mu^2)$

$$f = \Phi + \frac{\varepsilon}{2}(1 + 2\mu\eta)\Delta\Phi + \frac{5\varepsilon^2}{24}\Delta^2\Phi \tag{2.2}$$

Substituting (2.1) into (1.4) and (1.5), and using (2.2), we obtain the equations

$$\begin{gathered} \Phi_t + \eta + \frac{\mu}{2}(\nabla\Phi)^2 - \frac{\mu\varepsilon}{2}(\Delta\Phi)^2 = 0 \\ \eta_t + \operatorname{div}\left[(1 + \mu\eta)\nabla\Phi\right] + \frac{\varepsilon}{3}\Delta\left[(1 + 3\mu\eta)\Delta\Phi\right] + \frac{2\varepsilon^2}{15}\Delta^3\Phi = 0 \end{gathered} \tag{2.3}$$

where the divergence is expressed in terms of (x, y).

Thus, the functions η and Φ satisfy Eqs (2.3) and uniquely define the motion of the free surface. It has been shown [9] that η and Φ are canonical variables in the Hamiltonian description of waves on the water surface in the exact formulation; in other words, the motion of the free surface is described by the following equations in Hamiltonian form

$$\begin{gathered} \eta_t = \frac{\delta H}{\delta\Phi}, \quad \Phi_t = -\frac{\delta H}{\delta\eta} \\ H = \frac{1}{2}\int\int dxdy \left\{ \int_{-1}^{\mu\eta} \left[(\nabla\varphi)^2 + \frac{1}{\varepsilon}\varphi_z^2 \right] dz + \eta^2 \right\} \end{gathered} \tag{2.4}$$

where the integral is evaluated over the entire infinite volume of water. It can be shown using (1.2), (1.3), (2.1) and (2.2) that the Hamiltonian H can be transformed to

$$\begin{gathered} H = \frac{1}{2}\int\int dxdy \Big[\eta^2 - \Phi(1 + \mu\eta)\Delta\Phi - \mu\Phi\nabla\Phi\cdot\nabla\eta - \frac{\varepsilon}{3}\Phi\Delta^2\Phi - \\ - \mu\varepsilon\Phi\Delta(\eta\Delta\Phi) - \frac{2}{15}\varepsilon^2\Phi\Delta^3\Phi + O(\mu^3, \varepsilon^3) \Big] \end{gathered}$$

Computing the variation of H, we again obtain Eqs (2.3) from (2.4).

The system of equations derived in [10] for the potential at the free surface and its elevation up to cubic terms, unlike Eqs (2.3), was derived without any assumption as to the order of the characteristic scale of variation of the solution. This was done by introducing a special pseudodifferential operator. Benney's system for describing the interaction of four monochromatic waves was then extended to the case of a rough bottom and an arbitrary dependence of the initial wave form on the Cartesian coordinates, using the multidimensional WKB method (Maslov's operator method yields a global asymptotic expansion of rapidly oscillating solutions including caustic surfaces). In principle, Eqs (2.3) could have been obtained from system (3) of [10] by postulating a relationship between the dispersion and non-linearity parameters. Such an approach, however, would have been more complicated than that used in this paper.

Henceforth we will confine ourselves to the one-dimensional case, which represents long waves propagating in a straight channel with a smooth bottom. This means replacing ∇ by $\partial/\partial x$ and Δ by $\partial^2/\partial x^2$. Putting $V = \Phi_x$, we obtain from (2.3)

$$V_t + \eta_x + \mu V V_x - \mu\varepsilon V_x V_{xx} = 0$$

$$\eta_t + V_x + \mu(V\eta)_x + \frac{\varepsilon}{3} V_{xxx} + \frac{2}{15}\varepsilon^2 V_{xxxxx} + \mu\varepsilon\left(V_x\eta_{xx} + 2V_{xx}\eta_x + \eta V_{xxx}\right) = 0 \tag{2.5}$$

Obviously, instead of V one could introduce other variables characterizing the flow velocity, such as the velocity averaged over the depth, or the velocity at the bottom. The equations thus obtained are exactly similar to (2.5). This approximation has been justified for the shallow-water equations [8]. For that reason, Eqs (2.5) are asymptotically more accurate than the Boussinesq equations [7, 8], since the derivation incorporates higher-order terms in the asymptotic expansions. We would expect that, compared with the Boussinesq equations, they may describe shorter and more intense waves.

We can derive from (2.5) evolution equations for both V and η, by considering waves propagating in one direction from a source. To that end, let us assume that the functions V and η are related as follows:

$$V = \eta + c_1\mu\eta^2 + c_2\varepsilon\eta_{xx} + c_3\mu^2\eta^3 + c_4\varepsilon^2\eta_{xxxx} + c_5\mu\varepsilon\eta_x^2 + c_6\mu\varepsilon\eta\eta_{xx} \tag{2.6}$$

where the six constants c_i are to be determined. Considering (2.6) as a compatibility condition for Eqs (2.5), we can uniquely determine the c_is and obtain the following equation (an improved KdV equation)

$$\eta_t + \frac{3}{2}\mu\eta\eta_X + \frac{\varepsilon}{6}\eta_{XXX} + \frac{19}{360}\varepsilon^2\eta_{XXXXX} - \frac{3}{8}\mu^2\eta^2\eta_X + \frac{5}{12}\mu\varepsilon\eta\eta_{XXX} + \frac{23}{24}\mu\varepsilon\eta_X\eta_{XX} = 0 \tag{2.7}$$

where we have used the moving coordinate $X = x - t$. An evolution equation for V is obtained in similar fashion.

It turns out that there is a transformation that converts (2.7) into a completely integrable higher-order KdV equation, to within the same accuracy as the derivation of Eq. (2.7). This will be discussed in Section 3.

Incidentally, some of the results in [4, 5] are incorrect. First, the derivation of the system of equations of type (2.5) does not take all terms of order ε^2 into account. Second, when deriving an evolution equation from that system, it was assumed that

$$V = \eta + \mu A + \varepsilon B + \mu\varepsilon C + \varepsilon^2 D + \mu^2 E$$

where A, B, C, D and E are *functionally* dependent on η, and in order to eliminate derivatives with respect to t, the rule for differentiating composite functions $A_t = A_\eta\eta_t$, $B_t = B_\eta\eta_t$, etc. was used. However, in such cases of functional dependence, say $B = 1/6\eta_{xx}$, this rule is no longer sufficient. It is therefore necessary to assume a relationship of type (2.6), in which the *functional* dependences are explicitly represented. As that was not done in [4, 5], incorrect equations are obtained there for the elevation of the free surface η.

3. THE RELATIONSHIP BETWEEN EQS (2.7) AND A HIGHER-ORDER KdV EQUATION

It is well known [11, 12] that, besides the KdV equation, there is a class of equations, which we call here the KdV class, with the following properties. First, every equation of the class possesses a Hamiltonian structure and may be written as

$$u_t = \frac{\partial}{\partial x}\frac{\delta \tilde{h}}{\delta u}$$

where $\tilde{h}$ is the Hamiltonian of the given equation. Second, every Hamiltonian of the KdV class is an integral of motion for all the other equations of the class, and these integrals commute with one another relative to the Poisson bracket

$$\left\{\tilde{h}_1, \tilde{h}_2\right\} = \int_{-\infty}^{\infty} \frac{\delta \tilde{h}_2}{\delta u}\frac{\partial}{\partial x}\frac{\delta \tilde{h}_1}{\delta u}\, dx = 0$$

It is assumed, moreover, that the function u is rapidly decreasing, i.e. that it tends to zero sufficiently rapidly together with its derivatives as $x \to \pm\infty$. All equations of the KdV class, except for the KdV equation itself, are called higher-order KdV equations.

It can be shown that an equation

$$u_t + Euu_x + Fu_{xxx} + Au_{xxxxx} + Bu^2u_x + Cuu_{xxx} + Du_xu_{xx} = 0$$

belongs to the KdV class if $CE = 2BF$, $3CF = 5AE$, $D = 2C$.

We now consider Eq. (2.7) in the class of rapidly decreasing functions. It is remarkable that the transformation†

$$\eta = \zeta + a_1\varepsilon\zeta_{XX} + a_2\mu\zeta^2 + a_3\mu\zeta_X \int_{-\infty}^{X} \zeta dX \tag{3.1}$$

where a_1, a_2, a_3 are arbitrary constants of the order of unity, converts Eq. (2.7), accurately to within $O(\mu^2, \varepsilon^2)$, to the equation

$$\zeta_t + \frac{3}{2}\mu\zeta\zeta_X + \frac{1}{6}\varepsilon\zeta_{XXX} + \frac{19}{360}\varepsilon^2\zeta_{XXXXX} + \left(-\frac{3}{8} + \frac{3}{4}a_3 + \frac{3}{2}a_2\right)\mu^2\zeta^2\zeta_X + \left(\frac{5}{12} + \frac{1}{2}a_3\right)\mu\varepsilon\zeta\zeta_{XXX} + \left(\frac{17}{24} - 3a_1 + a_2 + \frac{1}{2}a_3\right)\mu\varepsilon\zeta_X\zeta_{XX} = 0 \tag{3.2}$$

The convergence of the integral in (3.1) is guaranteed by the assumption that ζ is a rapidly decreasing function. A transformation of type (3.1) was first pointed out by Kodama [13] in connection with KdV-type matrix equations.

Let us choose the constants a_1, a_2, a_3 so that Eq. (3.2) is a higher-order KdV equation. Solving a linear system of three equations for a_1, a_2, a_3, we determine them uniquely

$$a_1 = 1/2, \quad a_2 = 2, \quad a_3 = 3/4 \tag{3.3}$$

Thus, our problem of long-wave propagation on a water surface involves a higher-order KdV equation. It is obvious from (3.1) that the solution of Eq. (2.7) may be determined only to within $O(\mu, \varepsilon)$. Applying a scaling

$$\zeta = \frac{2}{3\mu}q, \quad t = 6\sqrt{\varepsilon}T, \quad X = \sqrt{\varepsilon}\chi \tag{3.4}$$

we rewrite the higher KdV equation (3.2), (3.3) in standard form

†It is erroneously stated in [3] that an equation of type (2.7) may be reduced to a higher-order KdV equation by (algebraic) point transformations.

$$q_\tau + 6qq_\chi + q_{\chi\chi\chi} + \frac{19}{360}\left(30q^2q_\chi + q_{\chi\chi\chi\chi\chi} + 20q_\chi q_{\chi\chi} + 10qq_{\chi\chi\chi}\right) = 0 \tag{3.5}$$

The Cauchy problem for Eq. (3.5) with initial condition $q(X, 0) = q_0(X)$, where $q_0(X)$ is a rapidly decreasing function, may be solved by the inverse scattering problem method [11, 12]. The components of the scattering matrix $a(k, T)$, $b(k, T)$ evolve in this case as follows:

$$a(k,T) = a(k,0), \quad b(k,T) = b(k,0)\exp\left[8ik^3T\left(1 - \frac{19}{15}k^2\right)\right]$$

where the initial data $a(k, 0)$, $b(k, 0)$ are determined by the asymptotic behaviour at $\pm\infty$ of the solution of Schrödinger's equation with potential $q_0(X)$

$$\psi_{XX} + \left(q_0 + k^2\right)\psi = 0$$

$$\psi(X) \sim e^{-ikX}, \quad X \to -\infty$$

$$\psi(X) \sim a(k,0)e^{-ikX} + b(k,0)e^{ikX}, \quad X \to +\infty$$

4. SOLITONS ON WATER

The simplest exact solution of Eq. (3.5) is the N-soliton solution, obtained for a reflection-less potential q (when $b(k, T) \equiv 0$). This solution may be written as

$$q(X,T) = 2\frac{d^2}{dx^2}\ln\det A \tag{4.1}$$

where A is a square matrix of order N with elements

$$A_{mn} = \delta_{mn} + \frac{c_m(T)c_n(T)}{\varkappa_m + \varkappa_n}\exp\left[-(\varkappa_m + \varkappa_n)X\right]$$

$$c_m(T) = c_m(0)\exp\left[4\varkappa_m^3T\left(1 + \frac{19}{15}\varkappa_m^2\right)\right]$$

δ_{mn} is the Kronecker delta, and $c_m(0)$ and $\varkappa_m$ are positive constants. Returning to the original variable η using (3.1), (3.3) and (3.4), we obtain the N-soliton solution of Eq. (2.7). The exact finite-band solutions are more complicated, as they are expressed in terms of the Riemann theta-function [11].

The single-soliton solution of Eq. (2.7) has the form

$$\eta(\xi) = \alpha\,\mathrm{sech}^2\,\xi + \frac{1}{2}\alpha^2\mu\,\mathrm{sech}^2\,\xi + \frac{3}{4}\alpha^2\mu\,\mathrm{sech}^4\,\xi + O(\mu^2) \tag{4.2}$$

$$\xi = \sqrt{\frac{3\alpha\mu}{4\varepsilon}}\left[X - \left(\frac{\alpha\mu}{2} + \frac{19}{40}\alpha^2\mu^2\right)t\right]$$

where $\alpha = 4\varkappa_1^2/3\mu \sim 1$ (the initial phase is omitted). The amplitude of the soliton (4.2) is $\alpha + 5/4\alpha^2\mu$ and its velocity in the moving system of coordinates is $1/2\alpha\mu + 19/40\alpha^2\mu^2$. Setting $\alpha = 1 - 5/4\mu$, we deduce from (4.2) that the velocity of a soliton with unit amplitude is $1/2\mu - 3/20\mu^2$, which is less than the velocity of a KdV soliton, while the width of the soliton in the improved theory exceeds that of a KdV soliton. This result was obtained by a different method in [1], where only steady solutions were considered. It is remarkable that the velocities of the solitons in the single- and double-soliton solutions (4.1), according to the improved theory, are in excellent agreement with the experimental data in [14].

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